when this is the case. This is a generalization of the well-known result that $\{\lambda\} = \{1\} \otimes (\lambda)$ vanishes if (λ) contains more than two nonzero parts. Thus, since the representation $\{\lambda_1, \lambda_2\}$ of the 2-dimensional unimodular unitary group equals $\{\lambda_1 - \lambda_2\}$, the analysis of $\{1\}^k$ is of the form $\{k\} + c_1\{k - 2\} + c_2\{k - 4\} + \ldots$, where $c_1 = k - 1$ is the dimension of the representation $(k - 1, 1)$ of the symmetric group on k symbols, $c_2 = k(k - 3)/2$ is the dimension of the representation $(k - 2, 2)$ of this symmetric group, and so on. We do not list $\{m\} \otimes (\lambda)$ when the number p of nonzero parts of (λ) is $m + 1$, for this is the same as $\{m\} \otimes (\lambda')$ where $(\lambda') = (\lambda_1 - \lambda_{m+1}, \lambda_2 - \lambda_{m+1}, \ldots, \lambda_m - \lambda_{m+1})$ is the partition, with less than $m + 1$ nonzero parts, of $k - (m + 1)\lambda_{m+1}$. For example, $\{2\} \otimes 1^3 = \{2\} \otimes 0 = \{0\}$; $\{2\} \otimes 321 = \{2\} \otimes 21$; $\{3\} \otimes 31^3 = \{3\} \otimes 2 = \{6\} + \{2\}$; and so on. We conclude with the remark that the relation $\{m\} \otimes 1^k = \{m - k + 1\} \otimes k$, together with the relation $\{m\} \otimes 1^k = \{m\} \otimes 1^{m-k+1}$ (which is an immediate consequence of the fact that $\{m\}$ is of dimension $m + 1$), imply the relation $\{m\} \otimes k = \{k\} \otimes m$, which is known as Hermite's Law of Reciprocity. We have given previously[1] another proof, based on the relation $(\{m - 1\} \otimes k)\{k - 1\} = (\{m\} \otimes k - 1)\{m - 1\}$, of this law.

I have to thank Professor E. P. Wigner for calling to my attention the importance in spectroscopy of the problem here discussed.

[1] F. D. Murnaghan, these PROCEEDINGS, **37**, 439–441, 1951; *Anais. Acad. Brasil. cienc.*, **23**, 347–368, 1951.

A VON STERNECK ARITHMETICAL FUNCTION AND RESTRICTED PARTITIONS WITH RESPECT TO A MODULUS

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Introduction.—Von Sterneck[1] in 1902 employed the number

$$\Phi(k, n) = \frac{\varphi(n)}{\varphi(n/(k, n))}\mu\frac{n}{(k, n)}, \tag{1}$$

where $\varphi(n)$ is the Euler indicator (or totient) and $\mu(n)$ is the Möbius number. Also, (k, n) is the greatest common divisor of k and n; k and n integers with $k \geq 0, n > 0$ and where $(0, n) = n$. (Further, unless otherwise stated, all small italic letters used will denote nonnegative rational integers.) He encountered (1) in investigating the number of different ways an integer s may be expressed as the sum of t integers if all the summands involved are reduced to their least residues modulo $m > 1$. He proved that

$$\sum_{d|n} \Phi(k, d) = \begin{cases} n \text{ if } (k, n) = n, \\ 0 \text{ otherwise.} \end{cases} \tag{1a}$$

He derived[2] also the relation, if $d|n$,

$$\sum_{k=1}^{n} \Phi(k, d) = \begin{cases} n \text{ if } d = 1, \\ 0 \text{ otherwise,} \end{cases} \tag{1b}$$

and noted the multiplicative property

$$\Phi(k, m)\Phi(k, n) = \Phi(k, mn); \qquad (m, n) = 1.$$

[Note that (1) reduces to $\varphi(n)$ when $(k, n) = n$ and to $\mu(n)$ when $(k, n) = 1$.] In the opinion of the authors, (1) will be very much employed by number theorists in the future and will eventually become classic in the literature of number theory; hence it will be convenient to give it a name. *We shall call it the "Von Sterneck number."*[3]

In 1920 Ramanujan[4] considered the number

$$C_n(k) = \sum_{(r, n) = 1} \alpha^{kr}; \qquad \alpha = e^{2i\pi/n}. \tag{2}$$

(Throughout this paper, α will have this meaning.) Such exponential expressions are called *Ramanujan sums.* He showed that

$$C_n(k) = \sum_{d|k,\, d|n} \mu(n/d)d. \tag{3}$$

He also employed $C_n(k)$ in certain infinite series.

In 1936 Hölder[5] *obtained the following fundamental result:*

$$C_n(k) = \Phi(k, n). \tag{4}$$

He gave two proofs of (4) and indicated another. He also obtained results concerning infinite series involving $C_n(k)$.

Nicol[6] obtained a number of relations involving (1). In particular he related $\Phi(k, n)$ to the theory of the restricted partitions of the natural numbers which have the two generating functions

$$\prod_{s=1}^{n} (1 \pm x^s). \tag{5}$$

Apostol and Anderson[7] defined a number which is a generalization of (1) and treated infinite series involving it. It is possible that some of the theorems of the present paper concerning (1) may be extended to the generalized number, but we have not yet examined this possibility.

Gagliardo[8] gave another proof of Hölder's theorem.

In the present paper we obtain a number of properties of (1) which involve the Φ's only, or the Φ's and roots of unity, the proofs all depending on the use of (4), excepting Theorem VIII. We then follow with an application of the same methods to a problem involving the partition of integers with respect to a modulus (Theorem VII).

This article is introductory to another article which we hope to publish in which arithmetical and analytic methods are applied to obtaining properties of the coefficients of (5) involving, usually, the Von Sterneck numbers.

Properties of the Φ-Function.—We shall have need in several places in this article of a result which we shall state as Lemma 1.

LEMMA 1. *If $d|n$, $c = \varphi(n)$ and*

$$r_1, r_2, \ldots, r_c \tag{6}$$

are the positive integers less than n and prime to n, then the number of the integers in *the set* (6) *which are congruent to t modulo* (n/d), *with* $(t, n/d) = 1$, *is* $(d/b)\varphi(b)$, *where* b *is the greatest divisor of* d *such that* $(b, n/d) = 1$.

Consider the set

$$t + j\frac{n}{d}, \tag{6a}$$

where $j = 0, 1, \ldots, d - 1$. The integers (6a) are prime to n/d but not necessarily prime to n. We shall select those which also have the latter property. Suppose that $b > 1$, where b is defined in the lemma. The d integers $0, 1, \ldots, d - 1$ may be written as

$$m + hb, \tag{7}$$

where $m = 0, 1, \ldots, b - 1$ and $h = 0, 1, \ldots, (d/b) - 1$. If we use these values of j in (6a) and reduce the resulting expressions modulo b, we obtain the set of residues

$$t + \frac{n}{d}m \tag{7a}$$

corresponding to each value of h. Since $(b, n/d) = 1$, this set modulo b reduces to a cyclic permutation of $0, 1, \ldots, b - 1$. By definition, $d = d_1b$, where each prime factor of d_1 divides n/d. To obtain the integers in (6a) which are prime to d and consequently prime to n, it is sufficient to select those which are prime to b, since they all prime to n/d and hence to d_1. This subset of (6a) reduces modulo b to the $\varphi(b)$ integers in the set $0, 1, \ldots, b - 1$ which are prime to b. Now there are d/b values of b corresponding to each set (7a). Hence the number of integers prime to n is $(d/b)\ \varphi(b)$, as stated in the lemma. If $b = 1$, each prime factor of d occurs in n/d, so that each element in (6) is prime to n. There are $(d/1)\varphi(1)$ or d such numbers, and the lemma also follows for this case.

As one application of this lemma, we shall give another[9] proof of Hölder's theorem. Suppose that a, r, and n are positive integers, and let

$$S = \sum_{(r,\, n)\, =\, 1} \alpha^{ar}, \qquad \text{where } \alpha = e^{2\pi i/n}.$$

Also, let $(a, n) = d$ and $kd = a$, so that $(k, n/d) = 1$. Denote the positive integers less than n and relatively prime to n by

$$r_1, r_2, \ldots, r_c, \qquad \text{where } c = \varphi(n),$$

and the positive integers less than n/d and relatively prime to n/d by

$$t_1, t_2, \ldots, t_v, \qquad \text{where } v = \varphi\left(\frac{n}{d}\right).$$

Then the exponents appearing in S are

$$kdr_1, kdr_2, \ldots, kdr_c. \tag{8}$$

Consider the set

$$t_i + j\frac{n}{d}, \tag{9}$$

where $i = 1, 2, \ldots, v$ and $j = 0, 1, \ldots, d - 1$. We then employ Lemma 1 and note that the number of integers prime to n for each t in (9) is the same. Altogether, there are $\varphi(n)$ positive integers less than n and prime to n. Consequently, (8) may be replaced by $kdt_1, kdt_2, \ldots, kdt_v$ repeated $\varphi(n)/\varphi(n/d)$ times. We shall assume the well-known relation[10]

$$\sum_{(r,\, n/d) = 1} \alpha^{dr} = \mu\left(\frac{n}{d}\right). \tag{9a}$$

Since $(k, n/d) = 1$, we have

$$\sum_{(r,\, n/d) = 1} \alpha^{kdr} = \mu\left(\frac{n}{d}\right). \tag{10}$$

Also, we note,

$$\sum_{(r,\, n) = 1} \alpha^{ra} = \sum_{(r,\, n) = 1} (\alpha^{d})^{ra/d}. \tag{11}$$

Then, from (10) and (11) and noting the remark just above (9a), we have

$$\frac{\varphi(n)}{\varphi(n/d)} \sum_{(r,\, n/d) = 1} \alpha^{kdr} = \frac{\varphi(n)}{\varphi(n/d)}\, \mu\left(\frac{n}{d}\right),$$

which proves (4).

We shall now prove (1a). (Von Sterneck's proof was more complicated.) Suppose d divides n and $n > 0$. Consider the set of positive integers less than $n + 1$ whose greatest common divisor with n is d. These may be written as $da_1, da_2, \ldots, da_v$, where $v = \varphi(n/d)$ and $a_1, a_2, \ldots, a_v$ are the $\varphi(n/d)$ integers less than n/d and prime to n/d. Then we have, by (4).

$$\sum_{d \mid n} \Phi(k, d) = \sum_{d \mid n} \sum_{(r,\, d) = 1} \alpha^{(n/d)rk},$$

since $\alpha^{n/d}$ is a dth root of unity. In the last relation, as d ranges over each divisor of n, we obtain the kth powers of all nth roots of unity, so that we may write

$$\sum_{d \mid n} \Phi\left(k, \frac{n}{d}\right) = 1 + \alpha^{k} + \alpha^{2k} + \ldots + \alpha^{(n-1)k},$$

and this sum is zero unless $\alpha^k = 1$, in which case it is n, which gives (1a).

If p is a prime such that $p = 1 + tn$, then the decomposition of (p) into prime ideal factors shows that one of them is $(g^t - 1, p)$, where g is a primitive root of p. Hence $\alpha \equiv g^t (\text{mod } \mathfrak{p})$, and, from (4), we have

$$\sum_{(r,\, n) = 1} g^{tkr} \equiv \Phi(k, n) \ (\text{mod } \mathfrak{p}),$$

or

$$\sum_{(r,\, n) = 1} g^{tkr} = \Phi(k, n) \ (\text{mod } p). \tag{12}$$

This proves a result due to Moller.[11]

Suppose $k > 0$, and consider the number

$$\sum_{s=1}^{n} \sum_{d|s} \Phi(k, s).$$

From relation (1a) we note that the inner sum is zero unless $(k, s) = s$ when it is s. Thus this number is the sum of the divisors of k less than or equal to n, which we shall denote by $\sigma(k, n)$.

However, we also have

$$\sum_{s=1}^{n} \sum_{d|s} \Phi(k, s) = \sum_{s=1}^{n} \left[\frac{n}{s}\right] \Phi(k, s),$$

where $[n/s]$ is the largest integer in n/s. Therefore, we may state.

THEOREM I. *We have the relation* (Paper N, [9]; *note also the two equations following it*)

$$\sum_{s=1}^{n} \left[\frac{n}{s}\right] \Phi(k, s) = \sigma(k, n), \tag{13}$$

where $k > 0$ and $\sigma(k, n)$ is the sum of the divisors of k less than or equal to n.

Consider the number

$$\sum_{t=0}^{(n/d)-1} \alpha^{-sdt} \Phi(dt, n).$$

Using the Hölder theorem, we have

$$\sum_{t=0}^{(n/d)-1} \alpha^{-sdt} \Phi(dt, n) = \sum_{t=0}^{(n/d)-1} \alpha^{-sdt} \sum_{(r, n)=1} \alpha^{rdt},$$

which may be written as

$$\sum_{(r, n)=1} (1 + \alpha^{d(r-s)} + \alpha^{2d(r-s)} + \ldots + \alpha^{((n/d)-1)(r-s)}).$$

The sum of the terms in the parenthesis is zero unless $\alpha^{(r-s)} = 1$, in which case it is n/d. Since α is a primitive nth root of unity, $\alpha^{(r-s)} = 1$ only if $r \equiv s \bmod (n/d)$; and the number of nonzero terms in the sum taken over the positive integers less than n and prime to n reduces to the number of positive integers less than n and prime to it which are congruent to s modulo (n/d), multiplied by n/d. If $(s, n/d) \neq 1$, then no r exists. If $(s, n/d) = 1$, then, by Lemma 1, the number of integers less than n and prime to n which are congruent to s modulo (n/d) is $(d/b)\varphi(b)$, where b is the largest divisor of d such that $(b, n/d) = 1$. This gives

THEOREM II. *If $n > 0$ and $d | n$, then*

$$\sum_{t=0}^{(n/d)-1} \alpha^{-tds} \Phi(dt, n) = \begin{cases} (n/b)\varphi(b) \text{ if } (s, n/d) = 1, \\ 0 \text{ otherwise,} \end{cases} \tag{14}$$

where $\alpha = e^{2\pi i/n}$ and b is the largest divisor of d such that $(b, n/d) = 1$.

Suppose that p is a prime and g is a primitive root modulo p. Then we shall show that, modulo p,

$$\sum_{t=1}^{p-1} \Phi(t, p-1)g^{ts} \equiv \begin{cases} -1 \text{ if } (s, p-1) = 1, & (15) \\ 0 \text{ otherwise.} & (16) \end{cases}$$

In relation (14), if we choose $d = 1$, then $b = 1$. Also select $n = p - 1$, where p is an odd prime with $(s, p - 1) = 1$. Then (14) becomes

$$\sum_{t=1}^{p-1} \alpha^{ts}\Phi(t, p-1) = p - 1. \tag{17}$$

If $\mathfrak{p}$ is a prime ideal divisor of (p), then we may write $\alpha \equiv g \pmod{\mathfrak{p}}$ for g a primitive root of p. Then (17) becomes

$$\sum_{t=1}^{p-1} g^{ts}\Phi(t, p-1) \equiv -1 \pmod{\mathfrak{p}}$$

or

$$\equiv -1 \pmod{p}, \text{ where } (s, p-1) = 1,$$

which is relation (15). In a similar way, relation (14), for $(s, p - 1) \neq 1$, becomes relation (16). Relations (15) and (16) may be combined to give the following results:

The only integral roots x of the congruence[6]

$$\sum_{t=1}^{p-1} x^t\Phi(t, p-1) \equiv -1 \pmod{p} \tag{18}$$

are the φ $(p - 1)$ incongruent primitive roots of p, and the only integral roots y of the congruence

$$\sum_{t=1}^{p-1} y^t\Phi(t, p-1) \equiv 0 \pmod{p} \tag{19}$$

are the integers in the set $0, 1, \ldots, p - 1$ which are not primitive roots of p.

THEOREM III. *If $n > 1$, then*

$$\sum_{s=1}^{n} s\Phi(s, n) = \frac{n\varphi(n)}{2}. \tag{20}$$

Proof: From (4) we have

$$\sum_{s=1}^{n} s\Phi(s, n) = \sum_{s=1}^{n} s \sum_{(r,n)=1} \alpha^{rs} = \sum_{(r,n)=1} (\alpha^r + 2\alpha^{2r} + \ldots + n\alpha^{nr}). \tag{20a}$$

As is well known, however,

$$\alpha^r + 2\alpha^{2r} + \ldots + n\alpha^{nr} = n\left(\frac{1}{\alpha^r - 1} + 1\right).$$

Consider now the function

$$F_n(x) = \prod_{(r,n)=1} (x - \alpha^r).$$

Differentiating and letting $x = 1$, we have

$$\frac{F_n'(1)}{F_n(1)} = -\sum_{(r,n)=1} \frac{1}{(\alpha^r - 1)}. \tag{20b}$$

Hölder[12] has shown that $F_n'(1)$ is equal to $(1/2)\varphi(n)$ when $n > 1$ and not a power of a prime, and $F_n'(1)$ is equal to $p^k(p-1)$ when $n = p^k$.
Also, as is well known,

$$F_n(1) = \begin{cases} 1 \text{ if } n \text{ contains two prime factors,} \\ p \text{ otherwise.} \end{cases}$$

Therefore, we have, from (20*a*) and (20*b*),

$$\sum_{s=1}^{n} s\Phi(s, n) = -\frac{n\varphi(n)}{2} + n\varphi(n),$$

which establishes the theorem.

If $n > 0$ and $2 | n$, consider the expression

$$S = 1 + \alpha^{2k} + \alpha^{4k} + \ldots + \alpha^{(n-2)k}. \tag{21}$$

Since α^2 is a $(n/2)$th root of unity, it follows that the above may be written, using (4), as

$$\sum_{d|n/2} \Phi\left(k, \frac{n}{2d}\right). \tag{22}$$

Now consider

$$\sum_{d|n} (-1)^d \Phi\left(k, \frac{n}{d}\right). \tag{23}$$

This may be written as $S - \alpha^k S$, using (21), since

$$\alpha^k + \alpha^{3k} + \ldots + \alpha^{(n-1)k} = \alpha^k S. \tag{24}$$

We also have, from (21), that $S = 0$, except when $k = n/2$, when it is $n/2$. Then $S - \alpha^k S$ reduces to 0, except when $k = n/2$, when it reduces to n, and we have[6] Theorem IV.

THEOREM IV. *If n is even and n and k are positive, then*

$$\sum_{d|n} (-1)^d \Phi\left(k, \frac{n}{d}\right) = \begin{cases} n \text{ if } k = n/2, \\ 0 \text{ otherwise.} \end{cases}$$

Let $\omega = e^{2\pi i/(n/d)}$ and $d|n$ and $d < n$. Then, by (4),

$$\sum_{s=1}^{n} \Phi\left(s, \frac{n}{d}\right) = \sum_{(r, n/d) = 1} \sum_{s=1}^{n} \omega^{rs} = d \sum_{s'=1}^{n/d} \omega^{rs'},$$

after we set $s = k(n/d) + s'$, with $0 \leq k < d$ and $0 \leq s' < n/d$. Hence the second factor on the right is zero. When $d = n$, the sum $\sum_{s=1}^{n} \Phi(s, 1) = n$. This gives (1*b*). Von Sterneck's proof was more complicated.

Consider the cyclotomic function

$$G(x) = \prod_{(r, n) = 1} (x - \alpha^r) \tag{25}$$

and the problem of finding a convenient way of determining its coefficients when written in polynomial form. First, the coefficients are, in absolute value, the elementary symmetric functions which are formed by the distinct $\varphi(n)$ primitive roots of unity. In view of (4), the sums of like powers of these roots are all known in terms of the Von Sterneck numbers.

Suppose $G(x) = x^c + p_1x^{c-1} + \ldots + p_c$ and $s_i = \sum_{(r,\,n)\,=\,1} \alpha^{ri} = \Phi\,(i, n)$, where $i = 1, 2, \ldots, c$; $c = \varphi(n)$. Consider the Newton formulas as applied to $G(x)$.

$$s_1 + p_1s_{i-1} + \ldots + p_{i-1}s_1 + ip_i = 0 \text{ for } i = 1, \ldots, c.$$

Let us assume that p_a $(a < c)$ has the property that it can be expressed as a polynomial in the Von Sterneck numbers with rational coefficients. Using the relation

$$s_{a+1} + p_1s_a + \ldots + (a + 1)p_{a+1} = 0,$$

it follows immediately by induction that p_{a+1} has the same property. Since this is obvious for p_1, it then follows that *each of the coefficients in the cyclotomic polynomial* (25) *may be expressed as a polynomial in the Von Sterneck numbers with rational coefficients.*

We shall now consider two inversion formulas.

THEOREM V. *If* $g_n(s) = \sum_{d|n} f(d)\Phi(s, n/d)$, *where* $f(n)$ *is an arithmetic function, then*

$$f(n) = \frac{1}{n}\sum_{s=1}^{n} g_n(s).$$

Proof: From the hypothesis we have

$$\sum_{s=1}^{n} g_n(s) = \sum_{s=1}^{n}\sum_{d|n} f(d)\Phi\left(s, \frac{n}{d}\right) = \sum_{d|n} f(d)\sum_{s=1}^{n}\Phi\left(s, \frac{n}{d}\right).$$

From relation (1*b*) the only term in the right-hand member which is not zero occurs when $d = n$, and thus the theorem follows.

THEOREM VI. *If* $g_n(d) = \sum_{k=1}^{n} f(k)\,\Phi(k, d)$, *where* $f(k)$ *is an arithmetic function, then*

$$f(n) = \frac{1}{n}\sum_{d|n} g_n(d).$$

The proof of this theorem is the same as in Theorem V, except that relation (1*a*) is used.

Restricted Partitions Modulo n.—Consider the expression

$$\sum_{t=0}^{n-1} \alpha^{-st}\Phi^m\,(t, n),$$

where m is a positive integer and α is a primitive nth root of unity. Then, from (4), we have

$$\sum_{t=0}^{n-1} \alpha^{-st}\Phi^m(t, n) = \sum_{t=0}^{n-1} \alpha^{-st}\sum_{(x_1,\,n)\,=\,1} \alpha^{tx_1}\ldots\sum_{(x_m,\,n)\,=\,1} \alpha^{tx_m} =$$
$$\sum_{(x_1,\,n)\,=\,1}\ldots\sum_{(x_m,\,n)\,=\,1}(1 + \alpha^y + \ldots + \alpha^{(n-1)y}),$$

where $y = (x_1 + x_2 + \ldots + x_m - s)$. The sum $1 + \alpha^y + \ldots + \alpha^{(n-1)y}$ is zero unless $y \equiv 0 \pmod n$ or $x_1 + \ldots + x_m \equiv s \pmod n$, in which case it is n. Then, if $P_m(m, s)$ is the number of solutions of the congruence $x_1 + \ldots + x_m \equiv s \pmod n$, in which the order of the terms is taken account of and $(x_i, n) = 1, i = 1, 2, \ldots, m$, we have

$$\sum_{t=0}^{n-1} \alpha^{-st}\Phi^m(t, n) = nP_n(m, s). \tag{26}$$

Suppose that d is a positive integer which divides n, and let S_d be the set of positive integers less than $n + 1$ whose greatest common divisor with n is d. Then we may write

$$\sum_{t=0}^{n-1} \alpha^{-st}\Phi^m(t, n) = \sum_{d|n} \sum_{S_d} \alpha^{-sd}\Phi^m(d, n),$$

where the inner sum is taken over the set S_d. However, for a positive integer D in the set S_d, the number $\Phi(D, n)$ remains constant, and we may write

$$\sum_{t=0}^{n-1} \alpha^{-st}\Phi^m(t, n) = \sum_{d|n} \Phi^m(d, n) \sum_{S_d} \alpha^{-sd}.$$

However, from (4), the inner sum may be replaced by $\Phi(s, n/d)$. Therefore we have Theorem VIII.

THEOREM VII.

$$\sum_{d|n} \Phi^m(d, n)\Phi\left(s, \frac{n}{d}\right) = nP_n(m, s). \tag{27}$$

where $P_n(m, s)$ is the number of solutions of the congruence $x_1 + x_2 + \ldots + x_m \equiv s \pmod n$, two solutions $(x_1, x_2, \ldots, x_m)$ and $(x_1', x_2', \ldots, x_m')$ being regarded as the same if and only if $x_i \equiv x_i' \pmod n$, where $(x_i, n) = 1$ for $i = 1, 2, \ldots, m$ and $m > 0, n > 1$.

For small values of m, at least, the right-hand member of (27) may be evaluated independently of the use of the Φ-number. For $m = 1$ in relation (26) we have a special case of Theorem II, when $d = 1$ in the latter statement. For $m = 2$ and $s = 0$ we determine explicitly the value of $P_n(m, s)$ as $\varphi(n)$. This[6] gives (26) for $m = 2, s = 0$. If we take $m = 3, s = 0$, then we have to find the x_i's with $(x_i, n) = 1, i = 1, 2, 3$, such that $x_1 + x_2 + x_3 \equiv 0 \pmod n$, which is equivalent to $x_1' + x_2' + 1 \equiv 0 \pmod n$, $(x_1', n) = 1$ and $(x_2', n) = 1$, so we have the problem of finding the number of sets of consecutive positive integers $< n$ and prime to n.

Schemmel[13] stated that $n \prod_p (1 - (2/p))$ is the number of pairs of consecutive positive integers less than n which are each prime to n. This is equivalent to the number of positive integers less than n which are relatively prime to n and less than n and which remain prime to n when unity is added to each of them. In this connection we have Theorem VIII.

THEOREM VIII. *If n is an integer greater than one, then*

$$\sum_{d|n} \Phi(d, n) = n \prod_p \left(1 - \frac{2}{p}\right),$$

where the product is taken over all prime divisors of n.

Proof: From relation (1), we have

$$\sum_{d|n} \Phi(d, n) = \varphi(n) \sum_{d|n} \frac{\mu(n/(d, n))}{\varphi(n/(d, n))}.$$

Since $d|n$, then $(d, n) = d$, and thus

$$\sum_{d|n} \Phi(d, n) = \varphi(n) \sum_{d|n} \frac{\mu(d)}{\varphi(d)}.$$

Since $\mu(d)$ and $\varphi(d)$ are multiplicative, then the number $\sum_{d|n} \Phi(d, n)$ is also multiplicative. Hence it is sufficient to consider $n = p^a$ where p is a prime and a is a positive integer. Then

$$\sum_{d|p^a} \Phi(d, p^a) = \varphi(p^a)\left(1 - \frac{1}{p - 1}\right) = p^a\left(1 - \frac{2}{p}\right),$$

which proves the theorem.

Von Sterneck[14] found an expression involving (1) for the number of representations of s as the residue modulo n of a sum of m elements chosen from 0, 1, . . . , $n - 1$, repetitions allowed but with order of the summands disregarded.

Whiteman[15] determined the number of sets of integers $u_1, u_2, \ldots, u_s$; $1 \leq u_i \leq m_i - 1$ for which $\alpha_1^{u_1} \ldots \alpha_s^{u_s} = 1$, where $\alpha_j = e^{2\pi i/m_j}$; $j = 1, 2, \ldots, s$. If m is the L.C.M. of these m_j's, this is the same as finding the number of representations $x_1, x_2, \ldots, x_s$ in $m_1x_1 + m_2x_2 + \ldots + m_sx_s \equiv 0 \pmod{m}$. This is equivalent to the problem[14] previously discussed, except that our representations are restricted to certain types of repetitions.

[1] *Sitzber. Akad. Wiss. Wien, Math.-Naturw. Klasse,* **111** (Abt. II*a*), 1567–1601, 1902. Cf. also Bachmann, *Niedere Zahlentheorie* (Leipzig: B. G. Teubner, 1909), **2,** 230–232.

[2] Bachmann, *op. cit.*, p. 232.

[3] We hope that other mathematicians will check our conclusion that Von Sterneck was the first investigator to isolate this number and obtain a few of its properties.

[4] *Trans. Cambridge Phil. Soc.*, **22,** 259–276, 1918; *Collected Papers*, pp. 179–199. To some extent these sums had been considered before; cf. Hardy and Wright, *The Theory of Numbers* (Oxford, 1938), p. 62.

[5] *Prace Mat.-Fiz.*, **43,** 13–23, 1936.

[6] These PROCEEDINGS, **39,** 963–968, 1953. This paper will be referred to as "Paper N." The present article is the result of joint work of Nicol and Vandiver, except as follows: Theorem I is due to Nicol (Paper N, [9]), who found a proof less direct than the one given here. He discovered Theorem IV and found a different proof. He obtained (26) for $m = 2$, $s = 0$ and obtained another type of proof for that case. The same remark applies to (18). All the results ascribed to Nicol were stated by him without proof in the reference given at the beginning of this footnote.

[7] D. R. Anderson and T. M. Apostol, *Duke Math. J.*, **20,** 211–216, 1953.

[8] E. Gagliardo, *Boll. Un. mat. Ital.*, (ser. 3), **8,** 269–273, 1953.

[9] Hölder gave two proofs of (4) and indicated another, and in addition we have another proof, as noted, by Gagliardo. If the reader desires to examine a proof of (4), our proof may be a convenience, as the articles of Hölder and Gagliardo were published in journals which are not readily accessible, at least in the United States.

[10] Cf. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, **2**, 572–573, 1909.
[11] *Am. Math. Monthly*, **59**, 228, 1953.
[12] *Op. cit.*, p. 15.
[13] *J. f. Math.*, **70**, 191–192, 1869. This and other of Schemmel's results were proved by Bachmann, *Niedere Zahlentheorie* (Leipzig: B. G. Teubner, 1902), **1**, 91–94.
[14] *Sitzber. Akad. Wiss. Wien, Math.-naturw. Klasse*, **114** (Abt. IIa), 711–730, 1905.
[15] *Trans. Am. Math. Soc.*, **74**, 87, relation (4.11), 1953.

SPHERICAL SPINORS IN A EUCLIDEAN 4-SPACE

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1. In a foregoing paper[1] some qualitative properties of baryons (nucleons and hyperons) in interaction with bosons were discussed in terms of a model in which these particles are characterized by quantum numbers corresponding to certain representations of the group of real 4-dimensional rotations with det. = +1. It is the purpose of this note to append these considerations by explicitly constructing the spinors which, in this model, describe the baryon states. This is a simple matter, and it is hoped that it may elucidate the group theoretical argument given previously.

It was important for the previous discussion[1] of charge-independence to note the connection between 3- and 4-dimensional rotations. This connection has long been known. In physical terms it is exhibited by the classical result[2] that the motion of a (3-dimensional) spherical top can be represented by the motion of a 4-dimensional spherical pendulum, i.e., a point moving over a 3-sphere in 4-space. Thus quantum-mechanically spherical top eigenfunctions can be written, in terms of appropriately chosen variables, as eigenfunctions of a particle with intrinsic spin zero constrained to a 3-sphere, i.e., as 4-dimensional spherical harmonics.[3] (Not all harmonics come into play, however; see the end of sec. 2.) Our present problem consists in constructing the eigenfunctions (spherical spinors) for a spin-$^1/_2$ particle subject to the same constraint.

In Section 2 the 4-dimensional spherical harmonics are expressed in a form somewhat different than usual,[4] so as to bring in evidence formal connections with the Legendre polynomials.[5] Recursion relations are obtained which closely resemble those holding for associated Legendre functions and by means of which it is easy to find the spherical spinors (sec. 3). Some features of the spinors in Euclidean 4-space have been studied by Schrödinger[6] in a little-known paper on the Dirac equation in a closed spherical universe. The procedure followed here is a different and rather simpler one.

Thus the main interest of this note is one of method. The results may perhaps be of use for field theoretical calculations, such as, for example, the relativistic two-body problem.

2. In a Euclidean 4-space (variables x_1, ..., x_4) we introduce the "angular momentum" operator